

Teleportation of general finite dimensional quantum systems

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Abstract

Teleportation of finite dimensional quantum states by a non-local entangled state is studied. For a generally given entangled state, an explicit equation that governs the teleportation is presented. Detailed examples and the roles played by the dimensions of the Hilbert spaces related to the sender, receiver and the auxiliary space are discussed.

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The discovery of quantum teleportation protocols belongs to the most important results of quantum information theory. Quantum teleportation has been introduced in [1] and discussed by a number of authors for both spin- $\frac{1}{2}$ states and arbitrary quantum states, see e.g. [2-11]. By means of a classical and a distributed quantum communication channel, realized by a non-local entangled state chosen in a special way (e.g., an EPR-pair of particles), the teleportation process allows to transmit an unknown quantum state from a sender (traditionally) named “Alice” to a receiver “Bob” that are spatially separated. For teleportation of N -dimensional quantum states, the teleportation problem has been discussed in [7] in the case where the dimensions of the Hilbert spaces associated with the sender, receiver and the auxiliary space are all equal to $N = 2^m$, for a given $m \in \mathbb{N}$. The relations among quantum teleportation, dense coding, orthonormal bases of maximally entangled vectors and unitary operators with respect to the Hilbert-Schmidt scalar product, and depolarizing operations are investigated in [8].

The details of protocols for teleportation vary with the shared entangled state and joint measurements at Alice or Bob. In this note we study the general properties of teleportation for finite dimensional quantum states without the assumption on equality for the dimensions of the Hilbert spaces involved. We give a teleportation protocol for generally given entangled states and a constraint equation that governs the teleportation. The solutions of the constraint equation give the unitary transformations of teleportation protocols. Detailed examples and the roles played by the dimensions of the Hilbert spaces are discussed.

Let H_1 be a Hilbert space with dimensions N_1 . Let e_i , $i = 1, \dots, N_1$, be an orthogonal basis in H_1 , so that e_i is an N_1 -dimensional column vector with entry 1 for the i -th component and 0 for the other components. Alice has a general quantum state on the Hilbert space H_1 of the form

$$\Psi_0 = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{N_1} \end{pmatrix} = \sum_{i=1}^{N_1} \alpha_i e_i, \quad (1)$$

where $\alpha_i \in \mathbb{C}$, $\sum_{i=1}^{N_1} |\alpha_i|^2 = 1$. For convenience we will call an arbitrary dimensional quantum state also a qubit in the following.

Let H_2 and H_3 be auxiliary Hilbert spaces attached to Alice and Bob, with dimensions N_2 and N_3 respectively. To send the state Ψ_0 to Bob’s hand, it is necessary that $N_3 \geq N_1$.

Let f_i (resp. g_j), $i = 1, \dots, N_2$ (resp. $j = 1, \dots, N_3$), be the corresponding orthogonal basis vector of the Hilbert space H_2 (resp. H_3). A generally entangled state of two qubits in the Hilbert spaces H_2 and H_3 is of the form

$$\Psi_1 = \sum_{i=1}^{N_2} \sum_{j=1}^{N_3} a_{ij} f_i \otimes g_j \quad (2)$$

for some (normalized) complex coefficients $a_{ij} \in \mathbb{C}$. The degree of entanglement depends on the a_{ij} , $i = 1, \dots, N_2$, $j = 1, \dots, N_3$. In the following we take $N_3 = N_1$.

The initial state Alice and Bob have is then given by

$$\Psi_0 \otimes \Psi_1 = \sum_{i,k=1}^{N_1} \sum_{j=1}^{N_2} \alpha_i a_{jk} e_i \otimes f_j \otimes g_k \in H_1 \otimes H_2 \otimes H_3. \quad (3)$$

Alice has the first and the second qubits and Bob has the third one. To transform the state of Bob's qubit to be Ψ_0 (given by (1)), one has to do some unitary transformation U and measurements. The effect of these operations together is called quantum "teleportation". Let U be the unitary transformation acting on the tensor product of two quantum states in the Hilbert spaces H_1 and H_2 such that

$$U(e_i \otimes f_j) = \sum_{s=1}^{N_1} \sum_{t=1}^{N_2} b_{ijst} e_s \otimes f_t, \quad (4)$$

with $\sum_{s=1}^{N_1} \sum_{t=1}^{N_2} |b_{ijst}|^2 = 1$, $\forall i = 1, \dots, N_1$, $j = 1, \dots, N_2$.

[Theorem]. If b_{ijst} satisfies the following relation

$$\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \alpha_i a_{jk} b_{ijst} = \frac{1}{\sqrt{N_1 N_2}} \alpha_{k-t+1} c_{s k-t+1 t} \quad (5)$$

for some $c_{ijk} \in \mathbb{C}$ such that $c_{ijk} c_{ijk}^* = 1$, U is the unitary transformation that fulfills the quantum teleportation.

[Proof]. From (4) and (5) we have, with Ψ_0 resp. Ψ_1 as in (1) resp. (2):

$$\begin{aligned} (U \otimes 1)(\Psi_0 \otimes \Psi_1) &\equiv \Phi = \sum_{i,s,k=1}^{N_1} \sum_{j,t=1}^{N_2} \alpha_i a_{jk} b_{ijst} e_s \otimes f_t \otimes g_k \\ &= \frac{1}{\sqrt{N_1 N_2}} \sum_{s,k=1}^{N_1} \sum_{t=1}^{N_2} \alpha_{k-t+1} c_{s k-t+1 t} e_s \otimes f_t \otimes g_k \\ &= \frac{1}{\sqrt{N_1 N_2}} \sum_{i,j=1}^{N_1} \sum_{k=1}^{N_2} c_{ijk} \alpha_j e_i \otimes f_k \otimes g_{k+j-1}, \end{aligned}$$

where the sub indices of e , f and g are understood to be taken modulo by N_1 , N_2 and N_3 respectively.

Now Alice measures her two qubits in the state $\Phi \in H_1 \otimes H_2$. If $e_i \otimes f_k$ is the state obtained after the measurement, i.e.,

$$\Phi \rightarrow e_i \otimes f_k \otimes \left(\sum_{j=1}^{N_1} c_{ijk} \alpha_j g_{k+j-1} \right),$$

then in order to recover the original state Ψ_0 , the unitary operator that Bob should use to act on his qubit is

$$O_{ik} = P_k C_{ik}, \quad i = 1, \dots, N_1, \quad k = 1, \dots, N_2, \quad (6)$$

where P_k is the $(k-1)$ -th power of the permutation operator, $P_k = \Pi^{k-1}$,

$$\Pi = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

and $C_{ik} = \text{diag}(c_{i1k}^*, c_{i2k}^*, \dots, c_{iN_1k}^*)$. After this transformation, one gets $\Phi \rightarrow e_i \otimes e_k \otimes \Psi_0$ and the state Ψ_0 given by (1) is teleported from Alice to Bob. ■

Therefore whenever an entangled state in the sense of (2) is given, i.e. the a_{ij} are given, if there are solutions of b_{ijst} to equation (5), we have a unitary transformation U that fulfills the teleportation. The condition (5) can be rewritten as

$$\sqrt{N_1 N_2} \sum_{i=1}^{N_2} a_{i,t+j-1} b_{ijst} = c_{sjt}. \quad (7)$$

The unitary transformation (4) given by the quantities b_{ijst} used in our teleportation protocol depends on the initially given entangled state (2) and the dimensions of the Hilbert spaces H_1 , H_2 , H_3 .

For general $N \equiv N_1 = N_2 = N_3$, if we take

$$a_{ij} = \frac{\delta_{ij}}{\sqrt{N}}, \quad (8)$$

the entangled state (2) is given by

$$\Psi_1 = \frac{1}{\sqrt{N}} \sum_{i=1}^N f_i \otimes g_i.$$

From equation (7), we obtain the unitary transformation (4) used in the teleportation protocol with

$$b_{i\ t+i-1\ st} = \frac{c_{sit}}{\sqrt{N}}, \quad (9)$$

with c_{sit} as in (5), the other coefficients b in (4) being zero. It is easily checked that the transformation (4) given by (9) is a unitary one. The teleportation is accomplished by applying the unitary operation (6) according to the result of Bob's measurement. For some particular values of the coefficients c_{sit} , this result coincides with the one in [11].

According to the Schmidt decomposition, in this case the entangled state (2) on Hilbert spaces H_2 and H_3 can be always written as

$$\sum_{i=1}^N \sqrt{\lambda_i} f_i \otimes g_i$$

in suitable basis, where $\lambda_i \geq 0$, $\sum_{i=1}^N \lambda_i = 1$. That is, $a_{ij} = \sqrt{\lambda_i} \delta_{ij}$. Substituting it into equation (7), we have

$$\sqrt{\lambda_{t+j-1}} b_{j\ t+j-1\ st} = c_{sit}.$$

According to the unitarity of the transformation U and the condition $c_{ijk} c_{ijk}^* = 1$, one gets $\lambda_i = 1/N$, $i = 1, \dots, N$, and the state (2) is a maximally entangled one, which shows that with a less than maximally entangled state it is impossible to give a unitary transformation that fulfills perfect teleportations.

For $N = 2$, taking $c_{111} = c_{211} = c_{121} = c_{112} = c_{212} = c_{122} = -c_{221} = -c_{222} = 1$ (this choice satisfies the condition $c_{ijk} c_{ijk}^* = 1$), we have $O_{11} = I$, $O_{12} = \sigma_x$, $O_{21} = \sigma_z$, $O_{22} = i\sigma_y$, where $\sigma_{x,y,z}$ are Pauli matrices and I is the 2×2 identity matrix. The unitary transformation U is then equal to the joint actions of the controlled-not gate C_{NOT} and the Walsh-Hadamard transformation H , as defined e.g. in [13, 14]. This recovers the usual protocol for teleporting two level quantum states given in [1].

When $N = 2^m$ for some $m \in \mathbb{N}$, a case discussed in [7], Ψ_1 can be rewritten as

$$\Psi_1 = \prod_{i=1}^m |EPR\rangle_i = \prod_{i=1}^m \frac{1}{\sqrt{2}} (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)_i,$$

where $|EPR\rangle = \frac{1}{\sqrt{2}} (|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$, an EPR pair of spin- $\frac{1}{2}$ particles, is the sum of all spin up and down states and $|EPR\rangle_i$ stands for the i -th EPR with the first (resp. second) attached to the Hilbert space H_2 (resp. H_3). Therefore instead of a fully entangled state of

two N -level qubits, we only need m pairs of entangled two-level qubits. This conforms with the discussions in [7].

Generally, the dimension N_2 of H_2 can be greater than N_1 . As long as one prepares the entangled state of two qubits in the Hilbert spaces H_3 and the sub Hilbert space $\mathcal{H}_2 \subset H_2$, with $\dim(\mathcal{H}_2) = N_1$, the above results are still valid. The entangled qubits attached to the Hilbert spaces H_2 and H_3 establish a quantum transportation “tunnel”, i.e. a way to teleport a qubit on H_1 to H_3 . To transport the qubit on H_1 to H_3 , this tunnel should be constructed in such a way that the entangled state is prepared with suitable coefficients a_{ij} . Moreover, this tunnel should be “broad” enough to let the quantum information go through, in the sense that the dimension N_2 of the Hilbert space H_2 should not be so small that there is no unitary transformation satisfying equation (5) for any kind of entangled state Ψ .

We consider now some special cases of teleportations when some components of the initial state are zero. Without losing generality, let $\alpha_i \neq 0$ for $i = 1, \dots, n_1$, $n_1 < N_1$, and $\alpha_i = 0$ for $i = n_1 + 1, \dots, N_1$ (We remark that for a given N_1 -dimensional vector it is always possible to make some of its components to be zero by changing the basis. But such a basis transformations depends of course on the components of the given vector, hence for an unknown quantum state this kind of transformation has no practical use).

The initial state to be teleported under the above hypothesis can be written as

$$\Psi_0 = \sum_{i=1}^{n_1} \alpha_i e_i.$$

We take the dimension of H_2 to be $N_2 = n_1 < N_1$. The entangled state used to teleport Ψ_0 can be prepared in the following way:

$$a_{ij} = \begin{cases} \frac{1}{\sqrt{n_1}} \delta_{ij}, & j = 1, \dots, n_1 \\ 0, & j = n_1 + 1, \dots, N_1 \end{cases} \quad (10)$$

for $i = 1, 2, \dots, n_1$. From (7) we get the unitary transformation (4) with

$$b_{i \ t+i-1 \ st} = \frac{c_{sit}}{\sqrt{N_1}}$$

for $t, t+i-1 \pmod{n_1} = 1, \dots, n_1$, $i, s = 1, 2, \dots, N_1$, the other coefficients b_{jist} in (7) being zero.

An example is the teleportation of an EPR pair $\Psi_0 = |\Psi_{EPR}\rangle = \alpha|01\rangle + \beta|10\rangle$, $|\alpha|^2 + |\beta|^2 = 1$, as discussed in [4]. In this case $N_1 = 4$. Ψ_0 can be written as $\alpha e_3 + \beta e_2 \equiv \alpha e'_1 + \beta e'_2$.

The dimension of the auxiliary Hilbert space H_2 is only needed to be $n_1 = 2$. The entangled state is given by $\Psi_1 = \frac{1}{\sqrt{2}}(f_1 \otimes g'_1 + f_2 \otimes g'_2) = \frac{1}{\sqrt{2}}(f_1 \otimes g_3 + f_2 \otimes g_2) = \frac{1}{\sqrt{2}}(|101\rangle + |010\rangle)$.

Here as $n_1 = N_1/2 = 2$, instead of (10), we may alternatively take $a_{ij} = \frac{1}{\sqrt{n_1}}\delta_{ij}$ for $j = n_1 + 1, \dots, N_1$ and $a_{ij} = 0$ for $j = 1, \dots, n_1$. Then the entangled state becomes $\Psi_1 = \frac{1}{\sqrt{2}}(f_1 \otimes g_1 + f_2 \otimes g_4) = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$, which is called a GHZ triplet consisting of three two-level qubits and can be realized experimentally [15, 16]. The unitary transformation is given by $b_{it+i-1st} = c_{sit}/\sqrt{N_1}$ for $t = 1, \dots, n_1$, $t + i - 1 \pmod{n_1} = n_1 + 1, \dots, N_1$, $i, s = 1, 2, \dots, N_1$, and the other coefficients b_{jist} in (7) being zero. For a suitable choice of the sign of c_{sit} , this recovers the result in [4].

We have studied the general properties of teleportation for finite dimensional discrete quantum states. The protocol we presented is for generally given entangled states with $N_3 = N_1$. If $N_3 > N_1$, one can always take a subspace $\mathcal{H}_3 \subset H_3$ such that $\dim(\mathcal{H}_3) = N_1$ and prepare the entangled state in the Hilbert spaces H_2 and \mathcal{H}_3 . Accordingly the initial state Ψ_0 will be sent to the subspace \mathcal{H}_3 .

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